Gulshan Kahtun

Practice Sheet #3
Continuity and Differentiability

(a) Test the continuity of the following functions:

1. $f(x)=\begin{cases}\cos x, & x\geq 0\\ -\cos x, & x<0\end{cases}$ at $x=0$.

2. $f(x)=\begin{cases}x\cos(1/x), x\neq 0\\ 0, \quad x=0\end{cases}$ at $x=0$.

3. $f(x)=\begin{cases}e^{1/x}, & x\neq 0\\ 1, & x=0\end{cases}$ at $x=0$.

4. $f(x)=\begin{cases}e^{\frac{-|x|}{2}}, & -1<x<0\\ x^2, & 0\leq x<2\end{cases}$ at $x=0$.

5. $f(x)=\begin{cases}(x-a)\sin\left(\dfrac{1}{x-a}\right), x\neq a\\ 0, \quad x=a\end{cases}$ at $x=a$.

6. $f(x)\begin{cases}1 & ,x<0\\ 1+\sin x & ,0\leq x<\pi/2\\ 2+(x-\pi/2)^2 & ,x\geq \pi/2\end{cases}$

at $x=0$ and $x=\pi/2$

7. $f(x)=\begin{cases}x\sin(1/x), x\neq 0\\ 0, \quad x=0\end{cases}$ at $x=0$.

8. $f(x)=\begin{cases}\dfrac{|x-3|}{x-3} & ,\cancel{x\neq 3}\ x\neq 3\\ 0 & ,\cancel{x=0}\ x=3\end{cases}$ at $x=3$ # $f(x)=\begin{cases}1-2x, & x<0\\ 1, & 0\leq x<1\\ 2x-1; & x>1\end{cases}$

9. ~~$f(x)=|x|+|x-1|$ at $x=0$ and $x=1$~~ at $x=0$ or, $x=1$.

~~10.~~ $f(x)=\begin{cases}(1+x)^{1/x} & ,x\neq 0\\ 1 & ,x=0\end{cases}$ at $x=0$. X

(b) Test the differentiability of the following functions:

1. $f(x)=\begin{cases}\cos x, & x\geq 0\\ -\cos x, & x<0\end{cases}$ at $x=0$.

3. $f(x)=|x|$ at $x=0$

2. $f(x)=\begin{cases}x^2\sin(1/x), x\neq 0\\ 0, \quad x=0\end{cases}$ at $x=0$.

4. $f(x)=\begin{cases}x\cos(1/x), x\neq 0\\ 0, \quad x=0\end{cases}$ at $x=0$.

5. Let $f(x)=\begin{cases}x^2-16x & ,x<9\\ 12\sqrt{x} & ,x\geq 9\end{cases}$. Is $f(x)$ continuous at $x=9$? Determine whether $f(x)$ is differentiable at $x=9$.

6. Let $f(x)=\begin{cases}x^2, & x\leq 1\\ \sqrt{x} & ,x>1\end{cases}$. Is $f(x)$ continuous at $x=1$? Determine whether $f(x)$ is differentiable at $x=1$.

7. Show that $f(x)=\begin{cases}x^2+1, & x\leq 1\\ x & ,x>1\end{cases}$ is not continuous and differentiable at $x=1$. Sketch the graph of $f(x)$. graph আঁকো

Shahrian

Practice Sheet # 3

(a) Continuity

1. $f(x) = \begin{cases} \cos x & ; x \geq 0 \\ -\cos x & ; x < 0 \end{cases}$ at $x = 0$

Solution:

(i) at $x = 0$

$f(0) = \cos 0 = 1$

(ii) $L.H.L = \lim_{x \to 0^-} f(x)$

$= \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} -\cos(-h)$

$= \lim_{h \to 0} -\cos h$

$= -\cos 0$

$= -1$

$R.H.L = \lim_{x \to 0^+} f(x)$

$= \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} \cos h$

$= \cos 0$

$= 1$

Since $L.H.L \neq R.H.L$

$\therefore f(x)$ is not continuous at $x = 0$. (Ans)

2. $f(x) = \begin{cases} x \cos(1/x) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ at $x = 0$

Solution:

(i) at $x = 0$

$f(0) = 0$

(ii) L.H.L $= \lim_{x \to 0^-} f(x)$

$= \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} -h \cos(-1/h)$

$= \lim_{h \to 0} -h \quad \lim_{h \to 0} \cos(1/h)$

$= -0 \cdot [-1 \le h \le 1]$

$= 0$

R.H.L $= \lim_{x \to 0^+} f(x)$

$= \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} h \cos(1/h)$

$= \lim_{h \to 0} h \quad \lim_{h \to 0} \cos(1/h)$

$= 0 \cdot [-1 \le h \le 1]$

$= 0$

Since L.H.L = R.H.L

$\therefore \lim_{x \to 0} f(x)$ = exists

(iii) $\lim_{x \to 0} f(x) = f(0)$

$\therefore f(x)$ is continuous at $x = 0$. (Ans)

3. $f(x) = \begin{cases} e^{1/x} & ; x \ne 0 \\ 1 & ; x = 0 \end{cases}$ at $x = 0$

Solution :

(i) at $x = 0$

$f(0) = 1$ c.

?

?

(ii) L.H.L $= \lim_{x \to 0^-} f(x)$

$= \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} e^{-1/h}$

$= \lim_{h \to 0} \frac{1}{e^{1/h}}$

R.H.L $= \lim_{x \to 0^+} f(x)$

$= \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} e^{1/h}$

$= \infty$

Since L.H.L $\neq$ R.H.L

$\therefore$ $f(x)$ is not continuous at $x=0$.

(Ans)

4. $f(x) = \begin{cases} e^{-\frac{|x|}{2}} & ; -1 < x < 0 \\ x^2 & ; 0 \leq x < 2 \end{cases}$ at $x=0$

Solution:

(i) at $x=0$

$f(0) = (0)^2 = 0$

(ii) L.H.L $= \lim\limits_{x\to 0^-} f(x)$

$= \lim\limits_{h\to 0} f(0-h)$

$= \lim\limits_{h\to 0} e^{\frac{-|-h|}{2}}$

$= \lim\limits_{h\to 0} e^{-h/2}$

$= e^{-0}$

$= 1$

R.H.L $= \lim\limits_{x\to 0^+} f(x)$

$= \lim\limits_{x\to 0} f(0+h)$

$= \lim\limits_{h\to 0} h^2$

$= 0$

Since L.H.L $\neq$ R.H.L

$\therefore$ $f(x)$ is not continuous at $x=0$.

(Ans)

5. $f(x) = \begin{cases} (x-a)\sin\left(\frac{1}{x-a}\right) & ; x \neq a \\ 0 & ; x = a \end{cases}$ at $x=a$

Solution:

(i) at $x=a$

$f(a) = 0$

(ii) $L.H.L = \lim_{x \to a^-} f(x)$

$= \lim_{h \to a} f(a-h)$

$= \lim_{h \to a} (a-h-a) \sin\left(\frac{1}{a-h-a}\right)$

$= \lim_{h \to a} -h \sin(-1/h)$

$= \lim_{h \to a} h \sin(1/h)$

$= 0 \cdot [-1 \leq h \leq 1]$

$= 0$

$R.H.L = \lim_{x \to a^+} f(x)$

$= \lim_{h \to a} f(a+h)$

$= \lim_{h \to 0} (a+h-a) \sin\left(\frac{1}{a+h-a}\right)$

$= \lim_{h \to a} h \sin(1/h)$

$= 0 \cdot [-1 \leq h \leq 1]$

$= 0$

Since $L.H.L = R.H.L$

$\therefore \lim_{x \to a} f(x) =$ exists

(iii) $\lim_{x \to a} f(x) = f(a)$

So $f(x)$ is continuous at $x = a$.

(Ans)

6. $f(x) = \begin{cases} 1 & ; x < 0 \\ 1 + \sin x & ; 0 \le x < \pi/2 \\ 2 + (x - \pi/2)^2 & ; x \ge \pi/2 \end{cases}$ at $x = 0$ and $x = \pi/2$

Solution: (part - I)

(i) at $x = 0$

$$f(0) = 1 + \sin 0$$
$$= 1$$

(ii) $\text{L.H.L} = \lim_{x \to 0^-} f(x)$

$$= \lim_{h \to 0} f(0 - h)$$
$$= \lim_{h \to 0} f(-h)$$
$$= 1$$

$\text{R.H.L} = \lim_{x \to 0^+} f(x)$

$$= \lim_{h \to 0} f(0 + h)$$
$$= \lim_{h \to 0} 1 + \sin h$$
$$= 1 + 0$$
$$= 1$$

Since L.H.L = R.H.L

$\therefore \lim_{x \to 0} f(x) =$ exists

(iii) $\lim_{x \to 0} f(x) = f(0)$

$\therefore f(x)$ is continuous at $x = 0$.

(Ans)

Solution: (Part - II)

(i) at $x = \pi/2$

$$f(\pi/2) = 2 + (\pi/2 - \pi/2)^2$$
$$= 2$$

(ii) $\text{L.H.L} = \lim_{x \to \pi/2^-} f(x)$

$$= \lim_{h \to 0} f(\pi/2 - h)$$
$$= \lim_{h \to 0} 1 + \sin(\pi/2 - h)$$
$$= \lim_{h \to 0} 1 + \text{[illegible]}\, h$$
$$= 1 + 1$$
$$= 2$$

$\text{R.H.L} = \lim_{x \to \pi/2^+} f(x)$

$$= \lim_{h \to 0} f(\pi/2 + h)$$
$$= \lim_{h \to 0} 2 + (\pi/2 + h - \pi/2)^2$$
$$= 2 + 0^2$$
$$= 2$$

Since L.H.L = R.H.L

$\therefore \lim_{x \to \pi/2} f(x) =$ exists

(iii) $\lim_{x \to \pi/2} f(x) = f(\pi/2)$

So, $f(x)$ is continuous at $x = \pi/2$

(Ans)

7. $f(x) = \begin{cases} x \sin(1/x) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ at $x = 0$

Solution:

(i) at $x = 0$

$f(0) = 0$

(ii) $L.H.L = \lim_{x \to 0^-} f(x)$

$= \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} -h \sin(-1/h)$

$= \lim_{h \to 0} h \sin(1/h)$

$= 0 \cdot [-1 \leq h \leq 1]$

$= 0$

$R.H.L = \lim_{x \to 0^+} f(x)$

$= \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} h \sin(1/h)$

$= 0. [-1 \leq h \leq 1]$

$= 0$

Since L.H.L = R.H.L

$\therefore \lim_{x \to 0} f(x) =$ exists.

(iii) $\lim_{x \to 0} f(x) = f(0)$

$\therefore f(x)$ is continuous at $x = 0$.

(Ans)

8. $f(x) = \begin{cases} \dfrac{|x-3|}{x-3} & ; x \neq 3 \\ 0 & ; x = 3 \end{cases}$ at $x = 3$

Solution:

(i) at $x = 3$

$f(3) = 0$

(ii) $L.H.L = \lim_{x \to 3^-} f(x)$

$= \lim_{h \to 0} f(3-h)$

$= \lim_{h \to 0} \dfrac{|3-h-3|}{3-h-3}$

$= \lim_{h \to 0} \dfrac{|-h|}{-h}$

$= \lim_{h \to 0} \dfrac{h}{-h}$

$= -1$

$R.H.L = \lim_{x \to {}^{+}} f(x)$

$= \lim_{h \to 0} f(3+h)$

$= \lim_{h \to 0} \dfrac{|3+h-3|}{3+h-3}$

$= \lim_{h \to 0} \dfrac{|h|}{h}$

$= \lim_{h \to 0} \dfrac{h}{h}$

$= 1$

Since, $L.H.L \neq R.H.L$

$\therefore \lim_{x \to 3} f(x)$ = does not exists

So, the function is not continuous at $x = 3$.

(Ans)

9. $f(x) = |x| + |x-1|$ at $x = 0$ and $x = 1$

$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

$|x-1| = \begin{cases} (x-1) & \text{if } x \geq 1 \\ -(x-1) & \text{if } x < 1 \end{cases}$

Solution: When $x < 0$;

then $|x| = -x$

and $|x-1| = -(x-1)$

$\therefore f(x) = -x - (x-1)$

$= -x - \quad 1$

When , $0 \le x < 1$;

then $|x| = x$

and $|x-1| = -(x-1)$

$$\therefore f(x) = x - (x-1)$$
$$= x - x + 1$$
$$= 1$$

When $x \ge 1$;

then $|x| = x$

and $|x-1| = (x-1)$

$$\therefore f(x) = x + x - 1$$
$$= 2x - 1$$

$$\therefore f(x) = \begin{cases} 1-2x & ; \; x < 0 \\ 1 & ; \; 0 \le x < 1 \\ 2x-1 & ; \; x \ge 1 \end{cases}$$ at $x = 0$ and $x = 1$

Solution (part-I) :

(i) at $x = 0$

$$f(0) = 1$$

(ii) $L.H.L = \lim_{x \to 0^-} f(x)$

$$= \lim_{h \to 0} f(0-h)$$
$$= \lim_{h \to 0} 1 + 2h$$
$$= 1$$

$R.H.L = \lim_{x \to 0^+} f(x)$

$$= \lim_{h \to 0} f(0+h)$$
$$= \lim_{h \to 0} 1$$
$$= 1$$

Since $L.H.L = R.H.L$

(iii) $\lim_{x \to 0} f(x) = f(0)$

so, $f(x)$ is continuous at $x = 0$. (Ans)

Solution : (part - II)

(i) at $x = 1$

$f(1) = 2 \cdot 1 - 1 = 1$

(ii) $L.H.L = \lim_{x \to 0^-} f(x)$

$= \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} 1$

$= 1$

$R.H.L = \lim_{x \to 0^+} f(x)$

$= \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} 2(1+h) - 1$

$= 2 + 0 - 1$

$= 1$

Since $L.H.L = R.H.L$

$\therefore \lim_{x \to 1} f(x) =$ exists.

(iii) $\lim_{x \to 1} f(x) = f(1)$

so, the $f(x)$ is continuous at $x = 1$. (Ans)

10. $f(x) = \begin{cases} (1+x)^{1/x} & ; x \neq 0 \\ 1 & ; x = 0 \end{cases}$ at $x = 0$

Solution :

(i) at $x = 0$

$f(0) = 1$

(ii) $f(x) = (1+x)^{1/x}$

$$= 1 + \frac{1}{x} \cdot x + \frac{1}{x} \cdot \left(\frac{1}{x} - 1\right) \cdot \frac{1}{2!} + \frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right) \cdot \frac{1}{3!} \cdot x^3 + \cdots$$

$$= 1 + \frac{1}{1!} + (1-x) \cdot \frac{1}{2!} + (1-x)(1 \quad 2x) \cdot \frac{1}{3!} + \cdots$$

$$\text{L.H.L} = \lim_{x \to 0^-} f(x)$$

$$= \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} \quad 1 + \frac{1}{1!} + \frac{1}{2!}(1+h) + \frac{1}{3!}(1+h)(1+2h) + \cdots$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$= e$$

$$\text{R.H.L} = \lim_{x \to 0^+} f(x)$$

$$= \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} \quad 1 + \frac{1}{1!} + \frac{1}{2!}(1-h) + \frac{1}{3!}(1-h)(1-2h) + \cdots$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$= e$$

Since L.H.L = R.H.L

$\therefore \lim_{x \to 0} f(x) =$ exists.

(iii) $\lim_{x \to 0} f(x) \neq f(0)$

$\therefore$ So, $f(x)$ is not continuous at $x = 0$.

(Ans)

Practice Sheet #3

(b) Differentiability

1. $f(x) = \begin{cases} \cos x & ; x \geq 0 \\ -\cos x & ; x < 0 \end{cases}$ at $x = 0$

Solution:

$$L.H.D = \lim_{h \to 0^-} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-\cos(-h) - \cos 0}{-h}$$

$$= \lim_{h \to 0} \frac{-(\cos h + 1)}{-h}$$

$$= \lim_{h \to 0} \frac{-\left(1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \cdots + 1\right)}{-h}$$

$$= \lim_{h \to 0} \frac{-2 + \frac{h^2}{2!} - \frac{h^4}{4!} + \cdots}{-h}$$

$$= \frac{2}{0} + 0 - 0 + \cdots$$

$$= \infty$$

$$R.H.D = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\cos h - 1}{h}$$

$$= \lim_{h \to 0} \frac{1 - \frac{h^2}{2!} + \frac{h^4}{4!} - \cdots - 1}{h}$$

$$= \lim_{h \to 0} \frac{-h}{2!} + \frac{h^3}{4!} - \cdots$$

$$= 0$$

Since L.H.D $\neq$ R.H

2. $f(x) = \begin{cases} x^2 \sin(1/x) & ; x \neq 0 \\ 0 & ; x = 0. \end{cases}$ at $x = 0$

Solution:

$$\text{L.H.D} = \lim_{h \to 0^-} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{(0-h)^2 \sin(0 - 1/h) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{-h^2 \sin(1/h)}{-h}$$

$$= \lim_{h \to 0} h \sin(1/h)$$

$$= 0$$

$$\text{R.H.D} = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{(0+h)^2 \sin\left(\frac{1}{0+h}\right) - 0}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin(1/h)}{h}$$

$$= \lim_{h \to 0} h \sin(1/h)$$

$$= 0$$

Since L.H.D = R.H.D

$\therefore f(x)$ is differentiable at $x = 0$.

(Ans)

3. $f(x) = |x|$ at $x=0$

Hence, $f(x) = \begin{cases} -x \; ; & x < 0 \\ x \; ; & x \geq 0 \end{cases}$

$$L.H.D = \lim_{h \to 0^-} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-(0-h) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h}$$

$$= -1$$

$$R.H.D = \lim_{h \to 0+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{(0+h) - 0}{h}$$

$$= \lim_{h \to 0} \frac{h}{h}$$

$$= 1$$

Since $L.H.D \neq R.H.D$

$\therefore f(x)$ is not differentiable at $x = 0$

(Ans)

1. $f(x) = \begin{cases} x\cos(1/x) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ at $x = 0$

Solution:

$$L.H.D = \lim_{h \to 0^-} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-h\cos\left(\frac{1}{-h}\right) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{-h\cos\left(\frac{1}{h}\right)}{-h}$$

$$= \lim_{h \to 0} \cos\left(\frac{1}{h}\right)$$

$$= \text{(undefined)}$$

$$R.H.D = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h\cos(1/h) - 0}{h}$$

$$= \lim_{h \to 0} \cos(1/h)$$

$$= \text{(undefined)}$$

Since $L.H.D \neq R.H.D$

$\therefore f(x)$ is not differentiable at $x = 0$.

(Ans)

Practice Sheet #3

(c) $f(x) = \begin{cases} x^2-16x \; ; & x < 9 \\ 12\sqrt{x} \; ; & x \geq 9 \end{cases}$ at $x = 9$

Solution: Continuity test:

(i) at $x = 9$

$f(9) = 12\sqrt{9} = 36$

(ii) $L.H.L = \lim_{x \to 9^-} f(x)$

$= \lim_{h \to 0} f(9-h)$

$= \lim_{h \to 0} (9-h)^2 - 16(9-h)$

$= 9^2 - 16 \cdot 9$

$= -63$

$R.H.L = \lim_{x \to 9^+} f(x)$

$= \lim_{h \to 0} f(9+h)$

$= \lim_{h \to 0} 12\sqrt{9+h}$

$= 12 \cdot \sqrt{9}$

$= 36$

Since $L.H.L \neq R.H.L$

$\therefore f(x)$ is not continuous at $x = 9$.

Differentiability test:

$L.H.D = \lim_{h \to 0^-} \frac{f(9-h) - f(9)}{-h}$

$= \lim_{h \to 0} \frac{(9-h)^2 - 16(9-h) - ...}{-h}$

$= \lim_{h \to 0} \frac{81 - 18h + h^2 - 14... + 16h - 36}{-h}$

$= \lim_{h \to 0} \frac{h^2 - 2h - 99}{-h}$

$= \lim_{h \to 0} \frac{-h(-h + 2 + \frac{99}{h})}{-h}$

$$R.H.D = \lim_{h\to 0^+} \frac{f(9+h) - f(9)}{h}$$

$$= \lim_{h\to 0} \frac{12\sqrt{(9+h)} - 36}{h} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{h\to 0} \frac{(12\sqrt{9+h} - 36)(12\sqrt{9+h} + 36)}{h(12\sqrt{9+h} + 36)}$$

$$= \lim_{h\to 0} \frac{144(9+h) - 144 \cdot 9}{h(12\sqrt{9+h} + 36)}$$

$$= \lim_{h\to 0} \frac{144(9+h-9)}{h(12\sqrt{9+h} + 36)}$$

$$= \lim_{h\to 0} \frac{144\cancel{h}}{\cancel{h}(12\sqrt{9+h} + 36)}$$

$$= 2$$

Since L.H.D $\neq$ R.H.D

$\therefore f(x)$ is not differentiable at $x = 9$. (Ans)

(d) $f(x) = \begin{cases} x^2 ; & x \leq 1 \\ \sqrt{x} ; & x > 1 \end{cases}$ at $x = 1$

Solution: Continuity test:

(i) at $x = 1$

$f(1) = (1)^2 = 1$

(ii) $L.H.L = \lim_{x\to 1^-} f(x)$

$= \lim_{h\to 0} f(1-h)$

$= \lim_{h\to 0} (1-h)^2$

$= (1)^2$

$R.H.L = \lim_{x\to 1^+} f(x)$

$= \lim_{h\to 0} f(1+h)$

$= \lim_{h\to 0} \sqrt{1+h}$

$= \sqrt{1}$

$\therefore \lim_{x \to 1} f(x) = \text{exists}$

(iii) $\lim_{x \to 1} f(x) = f(1)$

$\therefore f(x)$ is continuous at $x = 1$.

Differentiability test:

$$L.H.D = \lim_{h \to 0^-} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{(1-h)^2 - 1}{-h}$$

$$= \lim_{h \to 0} \frac{1 - 2h + h^2 - 1}{-h}$$

$$= \lim_{h \to 0} -h + 2$$

$$= 2$$

$$R.H.D = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{h \to 0} \frac{(\sqrt{1+h} - 1)(\sqrt{1+h} + 1)}{h(\sqrt{1+h} + 1)}$$

$$= \lim_{h \to 0} \frac{1 + h - 1}{h(\sqrt{1+h} + 1)}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{1+h} + 1)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{1+h} + 1}$$

Since L.H.D $\neq$ R.H.D

$\therefore f(x)$ is not differentiable at $x=1$. (Ans)

(e) $f(x) = \begin{cases} x^2+1 \; ; & x \leq 1 \\ x \; ; & x > 1 \end{cases}$ at $x=1$

Solution: Continuity test:

(i) at $x=1$

$f(1) = (1)^2+1 = 2$

(ii) L.H.L $= \lim_{x\to 1^-} f(x)$

$= \lim_{h\to 0} f(1-h)$

$= \lim_{h\to 0} (1-h)^2+1$

$= 1^2+1$

$= 2$

R.H.L $= \lim_{x\to 1^+} f(x)$

$= \lim_{h\to 0} f(1+h)$

$= \lim_{h\to 0} (1+h)$

$= 1$

Since L.H.L $\neq$ R.H.L

$\therefore f(x)$ is not continuous at $x=1$.

Differentiability test:

L.H.D $= \lim_{h\to 0^-} \dfrac{f(1-h) - f(1)}{-h}$

$= \lim_{h\to 0} \dfrac{(1-h)^2+1-2}{-h}$

$= \lim_{h\to 0} \dfrac{1-2h+h^2-1}{-h}$

$$= \lim_{h \to 0} 2 - h$$

$$= 2$$

$$R.H.D = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h) - 2}{h}$$

$$= \lim_{h \to 0} \frac{h-1}{h}$$

$$\boxed{= \frac{-1}{0} \\ = \infty}$$

Since L.H.D $\neq$ R.H.D

$\therefore f(x)$ is not differentiable at $x = 1$.

(Ans)