Assignment # 1

Maxima and minima

Last date of submission 18/02/08

1. Find (*a*) the open intervals on which f is increasing , (*b*) the open intervals on which f is decreasing, (*c*) the open intervals on which f is concave up, (*d*)) the open intervals on which f is concave down and (*e*) the x- coordinate of all inflection points.

   $(i)\ f(x) = x^2 - 5x + 6$ $\quad (ii)\ f(x) = 5 + 12x - x^3$

   $(iii)\ f(x) = x^4 - 8x^2 + 16$ $\quad (iv)\ f(x) = \dfrac{x^2}{x^2 + 2}$ $\quad (v)\ f(x) = \sqrt[3]{x + 2}.$

   $(vi)\ f(x) = Cosx;\ [0, 2\pi]$ $\quad (vii)\ f(x) = \tan x;\ [-\dfrac{\pi}{2}, \dfrac{\pi}{2}]$

2. Locate the critical numbers and identify which critical numbers correspond to stationary points.

   $(i)\ f(x) = x^3 + 3x^2 - 9x + 1$ $\quad (ii)\ f(x) = x^4 - 6x^2 - 3$

   $(iii)\ f(x) = \dfrac{x}{x^2 + 2}$ $\quad (iv)\ f(x) = x^{2/3}$

   $(v)\ f(x) = x^{1/3}(x + 4)$ $\quad (vi)\ f(x) = \cos 3x.$

3. Find the relative extrema (maxima/ minima) using both the first and second derivative tests.

   $(i) f(x) = 2x^3 - 9x^2 + 12x$ $\quad (ii) f(x) = \dfrac{x}{2} - \sin x, \quad 0 < x < 2\pi.$

4. Use the given derivative to find all critical numbers of f and determine whether a relative maximum, relative minimum, or neither occurs there.

   $(i)\ f'(x) = x^3(x^2 - 5)$ $\quad (ii)\ f'(x) = \dfrac{x^2 - 1}{x^2 + 1}.$

Maxima and minima

Solutions of the problems

1. (i) Given,

$$f(x) = x^2 - 5x + 6$$

$$\therefore f'(x) = 2x - 5$$

for the critical points

$$2x - 5 = 0$$

$$\therefore x = 5/2$$

intervals	$(2x-5)$	sign	conclusion
$(-\infty, 5/2)$	$0 <$	$-$	decreasing on $(-\infty, 5/2]$
$(5/2, \infty)$	$0 >$	$+$	increasing on $[5/2, \infty)$

$\therefore$ The function is decreasing on $(-\infty, 5/2]$ and the function is increasing on $[5/2, \infty)$.

$$f''(x) = 2$$

Here, $f''(x)$ = constant

So the concave up on $(-\infty, \infty)$ and no concave down.

The critical and stationary point is $x = 5/2$ and there is no inflection point.

1 (ii) Given,

$$f(x) = 5 + 12x - x^3$$

$$f'(x) = 12 - 3x^2$$

For the critical points

$$12 - 3x^2 = 0$$

$$\Rightarrow x^2 = 4$$

$$\therefore x = \pm 2$$

intervals	$12-3x^2$	sign	conclusion
$(-\infty, -2)$	$+$	$+$	increasing on $(-\infty, -2]$
$(-2, 2)$	$+$	$+$	increasing on $[-2, 2]$
$(2, \infty)$	$-$	$-$	decreasing on $[2, \infty)$

The function is increasing on $(-\infty, -2] \cup [-2, 2]$

And decreasing on $[2, \infty)$

$$f''(x) = -6x$$

$$-6x = 0$$

$$\therefore x = 0$$

intervals	$-6x$	sign	conclusion
$(-\infty, 0)$	$+$	$+$	concave up on $(-\infty, 0)$
$(0, \infty)$	$-$	$-$	concave down on $(0, \infty)$

$\therefore$ The Function is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$

The inflection point is 0

The critical and stationary point is ± 2

1.(III) Given,

$$F(x) = x^4 - 8x^2 + 16$$

$$\therefore F'(x) = 4x^3 - 16x$$

For the critical points

$$4x^3 - 16x = 0$$

$$\Rightarrow x^3 - 4x = 0$$

$$\Rightarrow x(x^2 - 4) = 0$$

$$\therefore x = 0, \pm 2$$

intervals	$4x^3 - 16x$	Sign	Conclusion
$(-\infty, -2)$	$-$	$-$	decreasing on $(-\infty, -2]$
$(-2, 0)$	$+$	$+$	increasing on $[-2, 0]$
$(0, 2)$	$-$	$-$	decreasing on $[0, 2]$
$(2, \infty)$	$+$	$+$	increasing on $[2, \infty)$

The function ~~θ~~ is decreasing on $(-\infty, -2] \cup [0, 2]$

And increasing on $[-2, 0] \cup [2, \infty)$

$$f''(x) = 12x^2 - 16$$

$$12x^2 - 16 = 0$$

$$\Rightarrow 3x^2 - 4 = 0$$

$$\Rightarrow x^2 = \frac{4}{3}$$

$$\therefore x = \pm \frac{2}{\sqrt{3}}$$

intervals	$12x^2-16$	sign	Conclusion
$(-\infty, -2/\sqrt{3})$	$+$	$+$	Concave up on $(-\infty, -2/\sqrt{3})$
$(-2/\sqrt{3}, 2/\sqrt{3})$	$-$	$-$	concave down on $(-2/\sqrt{3}, 2/\sqrt{3})$
$(2/\sqrt{3}, \infty)$	$+$	$+$	Concave up on $(2/\sqrt{3}, \infty)$

The function is concave up on $(-\infty, -2/\sqrt{3}) \cup (2/\sqrt{3}, \infty)$

And concave down on $(-2/\sqrt{3}, 2/\sqrt{3})$

The inflection points are $\pm 2/\sqrt{3}$

The critical and the stationary points are $0, \pm 2$

1 (iv) Given,

$$f(x) = \frac{x^2}{x^2+2}$$

$$f'(x) = \frac{(x^2+2)\cdot 2x - x^2 \cdot 2x}{(x^2+2)^2}$$

$$= \frac{2x^3 + 4x - 2x^3}{(x^2+2)^2}$$

$$= \frac{4x}{(x^2+2)^2}$$

For the critical points

$$\frac{4x}{(x^2+2)^2} = 0$$

$\therefore x = 0$

intervals	$\frac{4x}{(x^2+2)^2}$	Sign	Conclusion
$(-\alpha, 0)$	$-$	$-$	decreasing on $(-\alpha, 0]$
$(0, \alpha)$	$+$	$+$	increasing on $[0, \alpha)$

The function is decreasing on $(-\alpha, 0]$

And increasing on $[0, \alpha)$

$$f''(x) = \frac{(x^4+4x^2+4)\cdot 4 - 4x(4x^3+8x)}{(x^4+4x^2+4)^{\cancel{2}\,4}}$$

$$= \frac{4x^4+16x^2+16-16x^4-32x^2}{(x^4+4x^2+4)^{\cancel{2}\,4}}$$

$$= \frac{-12x^4-16x^2+16}{(x^4+4x^2+4)^{\cancel{2}\,4}}$$

$$\frac{-12x^{\cancel{2}\,4}-16x^2+16}{(x^4+4x^2+4)^2} = 0$$

$$\Rightarrow -12x^{\cancel{2}\,4}-16x^2+16=0$$

$$\Rightarrow 3x^4+4x^2 \mp 4 = 0$$

$$\Rightarrow 3x^4+6x^2-2x^2-4=0$$

$$\Rightarrow 3x^2(x^2+2)-2(x^2+2)=0$$

$$\Rightarrow (x^2+2)(3x^2-2)=0$$

$\therefore x^{2}\cancel{2}+2=0$ $\qquad\qquad$ $3x^2-2=0$

$\Rightarrow x^2=-2$ $\qquad\qquad$ $\Rightarrow x^2 = \frac{2}{3}$

$\therefore x = \sqrt{-2}$ $\qquad\qquad$ $\therefore x = \pm\sqrt{\frac{2}{3}}$

$\therefore x$ has no real value

intervals	$\frac{-12x^4-16x^2+16}{(x^2+2)^4}$	sign	conclusion
$(-\alpha, -\sqrt{\frac{2}{3}})$	$-$	$-$	concave down on $(-\alpha, -\sqrt{\frac{2}{3}})$
$(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}})$	$+$	$+$	concave up on $(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}})$
$(\sqrt{\frac{2}{3}}, \alpha)$	$-$	$-$	concave down on $(\sqrt{\frac{2}{3}}, \alpha)$

The function is concave down on $(-\alpha, -\sqrt{\frac{2}{3}}) \cup (\sqrt{\frac{2}{3}}, \alpha)$

And concave up on $(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}})$

The inflection points are $\pm\sqrt{\frac{2}{3}}$

The critical and stationary point is 0

1.(v) Given,

$$f(x) = \sqrt[3]{x+2}$$

$$\therefore f'(x) = \frac{1}{3}(x+2)^{1/3-1} \cdot 1$$

$$= \frac{1}{3}(x+2)^{-2/3}$$

For the critical points,

$$\frac{1}{3}(x+2)^{-2/3} = 0$$

$$\Rightarrow x+2 = 0$$

$\therefore x = -2$

intervals	$\frac{1}{3}(x+2)^{-2/3}$	Sign	Conclusion
$(-\infty, -2)$	$-$	$-$	decreasing on $(-\infty, -2]$
$(-2, \infty)$	$+$	$+$	increasing on $[-2, \infty)$

The function is decreasing on $(-\infty, -2]$

And increasing on $[-2, \infty)$

$$F''(x) = \frac{1}{3} \cdot -2/3 \ (x+2)^{-2/3-1} \cdot 1$$

$$= -\frac{2}{9}(x+2)^{-5/3}$$

$$-\frac{2}{9}(x+2)^{-5/3} = 0$$

$\therefore x = -2$

intervals	$-\frac{2}{9}(x+2)^{-5/3}$	Sign	Conclusion
$(-\infty, -2)$	$+$	$+$	Concave up on $(-\infty, -2)$
$(-2, \infty)$	$-$	$-$	concave down on $(-2, \infty)$

The function is concave up on $(-\infty, -2)$

And concave down on $(-2, \infty)$

The inflection point is -2

1.(vi) Given,

$$F(x) = \cos x \quad [0, 2\pi]$$

$f'(x) = -\sin x$

$-\sin x = 0$

$\Rightarrow \sin x = 0$

$\therefore x = 0, \pi, 2\pi$

intervals	$-\sin x$	sign	conclusion
$(0, \pi)$	$-$	$-$	decreasing on $[0, \pi]$
$(\pi, 2\pi)$	$+$	$+$	increasing on $[\pi, 2\pi]$

The function is decreasing on $[0, \pi]$

And increasing on $[\pi, 2\pi]$

$f''(x) = -\cos x$

$-\cos x = 0$

$\Rightarrow \cos x = 0$

$\therefore x = \pi/2, 3\pi/2$

intervals	$-\cos x$	sign	conclusion
$(0, \pi/2)$	$-$	$-$	concave down on $(0, \pi/2)$
$(\pi/2, 3\pi/2)$	$+$	$+$	concave up on $(\pi/2, 3\pi/2)$
$(3\pi/2, 2\pi)$	$-$	$-$	concave down on $(3\pi/2, 2\pi)$

The function is concave down on $(0, \pi/2) \cup (3\pi/2, 2\pi)$

And concave up on $(\pi/2, 3\pi/2)$

The inflection points are $\pi/2$ and $3\pi/2$

1. (vii) Given,

$$f(x) = \tan x \quad [-\pi/2, \pi/2]$$

$$f'(x) = \sec^2 x$$

$$\therefore \sec^2 x = 0$$

$$\Rightarrow \sec x = 0$$

$$\Rightarrow \frac{1}{\cos x} = 0$$

$$\Rightarrow \cos x = 0$$

$$\therefore x = -\pi/2, \pi/2$$

intervals	$\sec^2 x$	sign	conclusion
$(-\pi/2, \pi/2)$	$+$	$+$	increasing on $[-\pi/2, \pi/2]$

The function is increasing on $[-\pi/2, \pi/2]$

$$f''(x) = 2\sec x . \sec x . \tan x$$

$$= 2\sec^2 x . \tan x$$

$\therefore$ $2\sec^2 x . \tan x = 0$

$\Rightarrow \sec^2 x = 0$ and $\tan x = 0$

$\Rightarrow \frac{1}{\cos x} = 0$ $\therefore x = 0$

$\therefore x = -\pi/2, \pi/2$

intervals	$2\sec^2 x . \tan x$	sign	conclusion
$(-\pi/2, 0)$	$(+)$ $(-)$	$-$	concave down on $(-\pi/2, 0)$
$(0, \pi/2)$	$(+)$ $(+)$	$+$	concave up on $(0, \pi/2)$

The function is concave down on $(-\pi/2, 0)$

And concave up on $(0, \pi/2)$

The inflection point is 0

2(i) Given,

$$F(x) = x^3 + 3x^2 - 9x + 1$$

$$\therefore F'(x) = 3x^2 + 6x - 9$$

$$3x^2 + 6x - 9 = 0$$

$$\Rightarrow x^2 + 2x - 3 = 0$$

$$\Rightarrow x^2 + 3x - x - 3 = 0$$

$$\Rightarrow x(x+3) - 1(x+3) = 0$$

$$\Rightarrow (x+3)(x-1) = 0$$

$$\therefore x = -3, 1$$

For, $x = -3$

$$f'(x) = 3.(-3)^2 + 6.(-3) - 9$$
$$= \cancel{2}\,0$$

$x = 1$

$$f'(x) = 3.1^2 + 6.1 - 9$$
$$= 0$$

$\therefore x = -3, 1$ both are critical and stationary points.

2. (ii) Given,

$$f(x) = x^4 - 6x^2 - 3$$

$$f'(x) = 4x^3 - 12x$$

$4x^3 - 12x = 0$

$\Rightarrow x^3 - 3x = 0$

$\Rightarrow x(x^2 - 3) = 0$

$\therefore x = 0$ and $x^2 - 3 = 0$

$\therefore x = \pm\sqrt{3}$

at $x = 0$

$f'(x) = 0$

at $x = \sqrt{3}$

$f'(x) = 4.(\sqrt{3})^3 - 12\sqrt{3}$

$= 0$

at $x = -\sqrt{3}$

$f'(x) = 4(-\sqrt{3})^3 - 12(-\sqrt{3})$

$= 0$

$\therefore x = 0, \pm\sqrt{3}$ all are the critical and stationary points.

2. (iii) Given,

$$f(x) = \frac{x}{x^2+2}$$

$$\therefore f'(x) = \frac{(x^2+2) - x \cdot 2x}{(x^2+2)^2}$$

$$= \frac{x^2+2-2x^2}{(x^2+2)^2}$$

$$= \frac{2-x^2}{(x^2+2)^2}$$

Now, $\frac{2-x^2}{(x^2+2)^2} = 0$

$\Rightarrow 2 - x^2 = 0$

$\Rightarrow x^2 = 2$

$\therefore x = \pm\sqrt{2}$

at $x = \sqrt{2}$

$$f'(x) = \frac{2-(\sqrt{2})^2}{\{(\sqrt{2})^2+2\}^2}$$

$= 0$

at $x = -\sqrt{2}$

$$f'(x) = \frac{2-(-\sqrt{2})^2}{\{(-\sqrt{2})^2+2\}^2}$$

$= 0$

$\therefore x = \pm\sqrt{2}$ both are critical and stationary points.

2. (iv) Given,

$$F(x) = x^{2/3}$$

$$F'(x) = \frac{2}{3} x^{2/3 - 1}$$

$$= \frac{2}{3} x^{-1/3}$$

Now,

$$\frac{2}{3} x^{-1/3} = 0$$

$$\Rightarrow x^{-1/3} = 0$$

$$\therefore x = 0$$

at $x = 0$

$$F'(x) = \infty$$

$\therefore x = 0$ is the critical but not stationary point.

2(v) Given,

$$F(x) = x^{1/3}(x+4)$$

$$= x^{4/3} + 4x^{1/3}$$

$$F'(x) = \frac{4}{3} x^{4/3 - 1} + 4 \cdot \frac{1}{3} x^{1/3 - 1}$$

$$= \frac{4}{3} x^{1/3} + \frac{4}{3} x^{-2/3}$$

$= \frac{4}{3} x^{-1/3} (1 + x^{-1})$

Now,

$\frac{4}{3} x^{1/3} (1 + x^{-1}) = 0$

$\therefore x^{1/3} = 0$ and $1 + x^{-1} = 0$

$\therefore x = 0$

$\Rightarrow x + 1 = 0$

$\therefore x = -1$

$\therefore x = 0, -1$ are critical points

but $x = -1$ is only stationary points

2. (vi) Given,

$F(x) = \cos 3x$

$F'(x) = -3 \sin 3x$

Now,

$-3 \sin 3x = 0$

$\Rightarrow \sin 3x = 0$

$\Rightarrow 3x = n\pi$; $n = 0, \pm 1, \pm 2, \ldots\ldots$

$\therefore x = \frac{n\pi}{3}$

$\therefore\ x = \frac{n\pi}{3}$ are critical and stationary points.

3(i). Given,

$$F(x) = 2x^3 - 9x^2 + 12x$$

$$\therefore\ f'(x) = 6x^2 - 18x + 12$$

$f'(x) = 0$

$\Rightarrow 6x^2 - 18x + 12 = 0$

$\Rightarrow x^2 - 3x + 2 = 0$

$\Rightarrow x^2 - x - 2x + 2 = 0$

$\Rightarrow x(x-1) - 2(x-1) = 0$

$\therefore\ x = 1, 2$

$f''(x) = 12x - 18$

At $x = 1$,

$$F''(x) = 12 - 18$$
$$= -6$$

So, it has a maximum point at $x = 1$

The relative maximum value is,

$$2.1^3 - 9.1^2 + 12.1$$

At $x = 2$

$f''(x) = 24 - 18$

$= 6$

So, it has a minimum at $x = 2$

The relative minimum value is,

$2 \cdot 2^3 - 9 \cdot 2^2 + 12 \cdot 2$

$= 16 - 36 + 24$

$= 4$

4. (i) Given,

$f'(x) = x^3 (x^2 - 5)$

$f'(x) = 0$

$\Rightarrow x^3 (x^2 - 5) = 0$

$\therefore x^3 = 0$ or, $x^2 - 5 = 0$

$\therefore x = 0$ $x = \pm\sqrt{5}$

$f''(x) = 5x^4 - 15x^2$

at $x = 0$

$f''(x) = 0$

at $x = \sqrt{5}$

$F''(x) = 50$

at $x = -\sqrt{5}$

$F''(x) = 50$

so, the Function has a minimum at point $x = \pm\sqrt{5}$ and the function has neither a maximum or nor a minimum at point $x = 0$

4. (ii) Given,

$$f'(x) = \frac{x^2-1}{x^2+1}$$

$f'(x) = 0$

$\Rightarrow \frac{x^2-1}{x^2+1} = 0$

$\Rightarrow x^2 - 1 = 0$

$\therefore x = \pm 1$

so. the critical numbers are ± 1

$$f''(x) = \frac{(x^2+1)\cdot 2x - (x^2-1)\cdot 2x}{(x^2+1)^2}$$

$$= \frac{2x^3 + 2x - 2x^3 + 2x}{(x^2+1)^2}$$

$$= \frac{4x}{(x^2+1)^2}$$

at $x=1$

$$f''(1) = \frac{4}{4}$$

$$= 1$$

So. it han a minimum at point $x=1$

at $x=-1$

$$f''(-1) = \frac{-4}{4}$$

$$= -1$$

So, it han a maximum at point $x=-1$